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TRACK FITTING WITH MULTIPLE SCATTERING : A NEW METHOD

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ABSTRACT :

An analytical calculation of the variance is performed, in some simple cases, for standard least-squares estimators of track parameters (accounting for independent measurement errors only) ; comparison is made with optimal estimators (accounting also for scattering errors, correlated between one point and the following ones). A new method is proposed for optimal estimation : the points measured on the track are included backwards, one by one, in the fitting algorithm, and the scattering is handled locally at each step. The feasability of the method is shown on real events, for which the geometrical resolution is improved. The algorithm is very flexible and allows fast programmation ; moreover the computation time is merely proportional to the number of measured points, contrary to the other optimal estimators.

RESUME :

Les variances des estimateurs standard par moindres carrés des paramètres d'une trace (qui ne tiennent compte que d'erreurs de mesure indépendantes) sont calculées analytiquement dans quelques cas simples, et comparées à celles des estimateurs optimaux (qui tiennent compte aussi des erreurs dues à la diffusion, corrélées entre un point et les suivants). Une nouvelle méthode est proposée pour l'estimation optimale : les points mesurés sur la trace sont inclus l'un après l'autre, à partir du dernier, dans l'algorithme d'ajustement, et la diffusion est traitée localement à chaque étape. La faisabilité de la méthode est démontrée sur des événements réels, pour lesquels la résolution géométrique est améliorée. L'algorithme est très souple et se prête à une programmation rapide ; en outre le temps de calcul est simplement proportionnel au nombre de points mesurés, contrairement aux autres estimateurs optimaux.

1. INTRODUCTION :

Charged particles going through matter are affected by random deviations due to multiple scattering. The uncertainty on their initial fitted parameters (position and momentum components) arises form two contributions : on the one hand the contribution of the measurement errors, which is a decreasing function of the number of measured points ; on the other hand the contribution of the multiple scattering errors, which cannot be reduced below some minimum values, because the detectors contain at least a gas at atmospheric pressure, and some denser parts. These scattering uncertainties become predominant for low momentum particles, or for very accurate measurements.

There are essentially two ways to perform a geometrical track fit when the scattering errors are not negligible :

1) A standard fit takes only the measurement errors into account, and provides estimators for the track parameters and their covariance matrix (as resulting from the measurement errors only). Such estimators are not optimal (some information is lost). Their actual covariance matrix can be obtained by adding afterwards the multiple scattering contribution.

2) An optimal fit makes use of the full (n x n) covariance matrix of the n measurements (including multiple scattering, i.e. correlation terms). This is usually realized by using the Gauss-Markow theorem, and needs then the inversion of this matrix (1) .

The precision of the estimators corresponding to both methods was already compared by Drijard (2) by numerical evaluation, for the curvature and angle parameters only. In this paper we give analytical expressions of the covariance matrices, and we consider also position parameters, because the kinematical resolution depends on the precision on these parameters when the tracks are extrapolated to a vertex.

In sect. 2 we calculate the matrices resulting from the standard method in some simple cases, and we show that an increase of the number of measured points may lead to an increasing error on the parameters ; in sect. 3 we describe a recursive solution of the optimal method, hopefully less expensive than the brute-force matrix inversion involved in the Gauss-Markov method ; in sect. 4 we determine recursively the optimal covariance matrix of the parameters in the same cases as in sect. 2 in order to evaluate the gain in precision. In sect. 5 we present results obtained with real events from OMEGA Spectrometer at C.E.R.N.

Let us recall the expression of the variance of the projected scattering angle for a particle of charge Z , velocity v, momentum p, after passing through ξ radiation lengths :

$$\Delta\alpha^2 = \left(\frac{K.Z}{p.v}\right)^2 \xi$$

where K is about 15 MeV (with a correcting factor at low velocities).

The main point for our purpose is that for a given particle with a given momentum, $\Delta\alpha^2$ is proportional to ξ .

2. THE STANDARD FIT : EVALUATION OF THE ERROR MATRIX

2.1. General conventions :

We suppose that all measured points on the track are equidistant (interval ℓ), with the same precision, and that their number n is big enough to replace the summations over the points by integrations (or equivalently to hold only the terms of highest degree in n).

We consider slightly curved tracks in an uniform vertical magnetic field : in the absence of errors they would have a straight vertical projection and a parabolic horizontal projection. These conditions are often realistic, and our conclusions hold qualitatively in more general cases.

In order to simplify the expressions, we replace the measured coordinates by their deviations from the theoretical ones (which would be obtained without errors and scattering). If the estimators are not biased, the mean value of the fitted parameters will then vanish, and their variances will come to their mean squared values.

2.2. Scattering at one point of the track :

2.2.1. General notations :

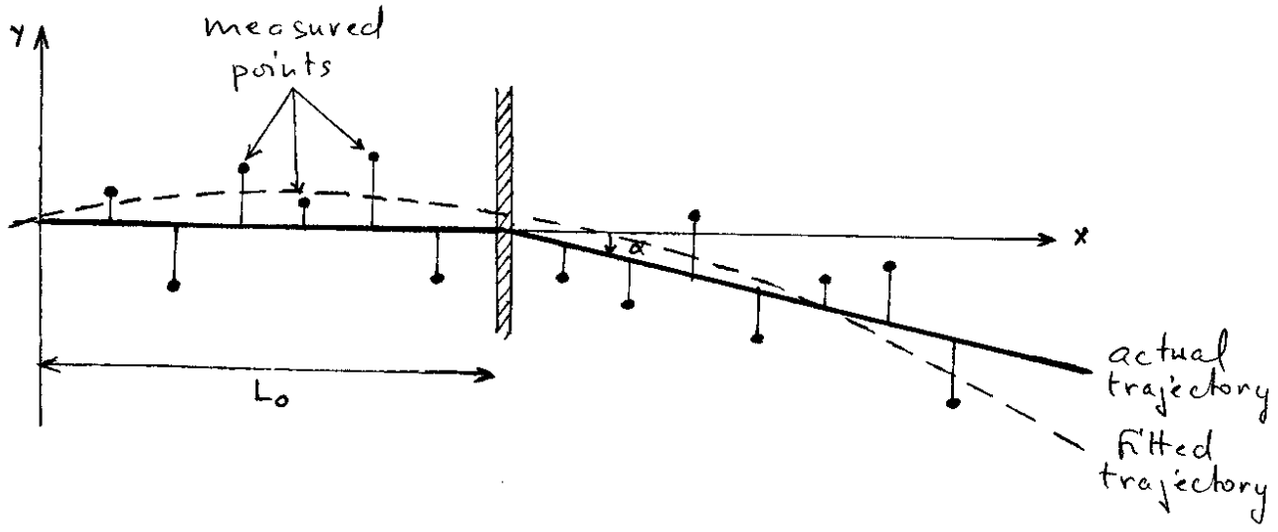
We suppose that the particle encounters matter concentrated at a distance L_o after the first measured point. Let $n_o = L_o/\ell$ be the number of points before L_o , and $\Delta\alpha^2 = (\frac{K.Z}{p.v})^2 \xi$ the variance of the scattering angle α in both vertical and horizontal directions.

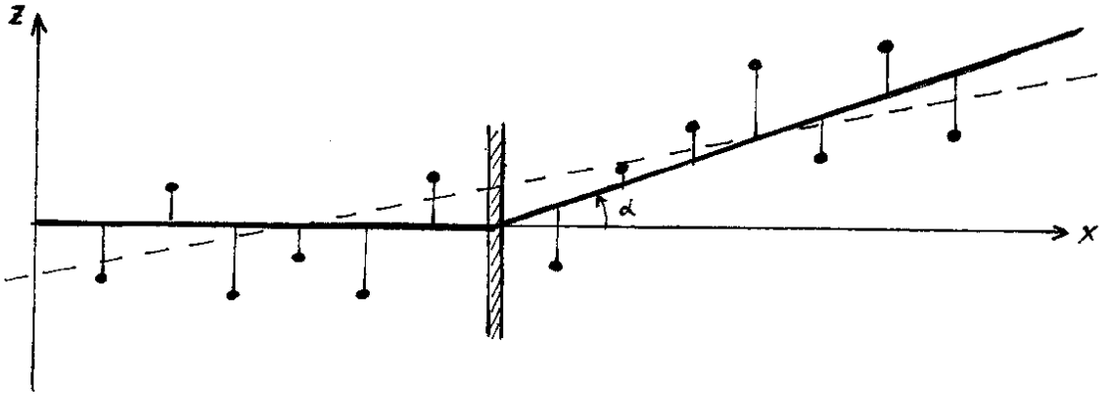
The measured coordinates at abscissa $x_i = i\ell$ will be called y_i^m for the horizontal projection (parabolic fit) and z_i^m the vertical one (straight fit). They can be expressed by :

$$\begin{cases} y_i^m = \delta y_i & \text{for} \quad i \leqslant n_o \\ y_j^m = \delta y_j + (j-n_o)\,\ell\,\alpha & \text{for} \quad j > n_o \end{cases} \tag{2.1}$$

and similarly for z_i^m .

δy_i and δz_i are random variables of variance σ^2. α is a variable of variance $\Delta\alpha^2$. All of them are independent.





The standard fit determines on the one hand the parameters Z and b (position at the first measured point and slope in the vertical plane), and on the other hand Y , a and c = $\frac{1}{R}$ (position and slope at the first point in the horizontal plane, and curvature), as linear combinations of the measured coordinates, hence of the independent variables δz_i (or δy_i) and α .

Thus the variances of these parameters are the sum of the measurement errors contribution and the scattering contribution, and we can calculate them separately. The same thing is true for the sum of squared residuals, and then for the χ^2.

2.2.2. Straight line fit :

From the equation of the trajectory :

$$z = Z + b\,x$$

the parameters Z and b are given by the matrix equation :

$$\begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix} \begin{bmatrix} Z \\ b \end{bmatrix} = \begin{bmatrix} \sum_i z_i^m \\ \sum_i x_i\, z_i^m \end{bmatrix} \qquad (2.2)$$

The inverse of the left-hand side matrix, multiplied by σ^2 , is the covariance matrix V^m of the parameters in the absence of scattering :

$$V^m \simeq \sigma^2 \begin{bmatrix} n & \frac{n^2}{2}\ell \\ \frac{n^2}{2}\ell & \frac{n^3}{3}\ell^2 \end{bmatrix}^{-1} = 12\,\sigma^2 \begin{bmatrix} \frac{1}{3n} & -\frac{1}{2n^2\ell} \\ -\frac{1}{2n^2\ell} & \frac{1}{n^3\ell^2} \end{bmatrix}$$

or, with $L=n\ell$ (total measured length) :

$$V^m_{ZZ} = \frac{4\sigma^2}{n} \quad , \quad V^m_{bb} = \frac{12\sigma^2}{nL^2} \quad , \quad V^m_{Zb} = -\frac{6\sigma^2}{nL} \tag{2.3}$$

The χ^2 has a well known mean value

$$\langle\chi^2\rangle^m = n-2 \simeq n$$

If we consider now the α variable only, we obtain for the right-hand side of (2.2) :

$$\sum_i z_i^m = \ell\,\alpha \sum_{i>n_o} (i-n_o) \simeq \frac{(n-n_o)^2}{2}\,\ell\,\alpha$$

$$\sum_i x_i z_i^m = \ell^2\,\alpha \sum_{i>n_o} i(i-n_o) \simeq \frac{(n-n_o)^2(2n+n_o)}{6}\,\ell^2\,\alpha$$

(with, of course, $n > n_o$)

whence Z and b as functions of α :

$$Z = \frac{4}{n}\,\ell\,\alpha\,\frac{(n-n_o)^2}{2} - \frac{6}{n^2\ell}\,\ell^2\alpha\,\frac{(n-n_o)^2(2n+n_o)}{6} = -\frac{n_o(n-n_o)^2}{n^2}\,\ell\,\alpha$$

$$b = -\frac{6}{n^2\ell}\,\ell\alpha\,\frac{(n-n_o)^2}{2} + \frac{12}{n^3\ell^2}\,\ell^2\alpha\,\frac{(n-n_o)^2(2n+n_o)}{6} = \frac{(n+2n_o)\,(n-n_o)^2}{n^3}\,\alpha$$

and their covariance matrix :

$$V^s_{ZZ} = \frac{n_o^2(n-n_o)^4}{n^4}\,\ell^2\,\Delta\alpha^2 \quad , \quad V^s_{bb} = \frac{(n+2n_o)^2(n-n_o)^4}{n^6}\,\Delta\alpha^2$$
$$V^s_{Zb} = -\frac{n_o(n+2n_o)(n-n_o)^4}{n^5}\,\ell\,\Delta\alpha^2 \tag{2.4}$$

The variance of any quantity depending on Z and b is a linear combination of the elements of the matrix $V^t = V^m + V^s$

If we assume n_o and ℓ to be fixed (it is true for almost all detectors, execpt for bubble chambers) we are led to study the dependence of V^t_{ZZ} and V^t_{bb} on n (or L) to evaluate the precision of the standard estimator as a function of the measured length :

$$V^t_{ZZ} = \frac{4}{n}\,\sigma^2 + \frac{n_o^2(n-n_o)^4}{n^4}\,\ell^2\,\Delta\alpha^2$$

$$V^t_{bb} = \frac{12}{n^3\ell^2}\,\sigma^2 + \frac{(n+2n_o)^2(n-n_o)^4}{n^6}\,\Delta\alpha^2$$

The σ^2 terms decrease and vanish for $n \to \infty$.
The $\Delta\alpha^2$ terms increase and tend to finite limits $V_{ZZ}^{\infty} = n_o^2\, \ell^2\, \Delta\alpha^2 = L_o^2\, \Delta\alpha^2$ and $V_{bb}^{\infty} = \Delta\alpha^2$, which were to be foreseen, because for infinite n the parameters are determined from the second part of the track (after L_o) with an infinite precision.

Now, in order to look for the best precision available on the parameters with the standard estimators, let us see whether the total variances go through a minimum as n varies. With

$$\rho^2 = \frac{n_o^3\, \ell^2\, \Delta\alpha^2}{\sigma^2} = \frac{L_o^2\, \Delta\alpha^2}{\sigma^2/n_o} \qquad (2.5)$$

one obtains the following conditions :

$$\text{for } V_{ZZ}^t : \quad \frac{(n-n_o)^3}{n^3} = \frac{1}{\rho^2}$$

$$\text{for } V_{bb}^t : \quad \frac{(n+2n_o)(n-n_o)^3}{n_o n^3} = \frac{1}{\rho^2}$$

Thus V_{bb}^t has always a minimum, and V_{ZZ}^t has a minimum for $\rho > 1$.

The dimensionless quantity ρ can be interpreted as the ratio of $L_o\Delta\alpha$, scattering error propagated over L_o to the first point, over $\frac{\sigma}{\sqrt{n_o}}$, measurement error statistically compensated on n_o points.

It is interesting to calculate an order of magnitude of ρ in current experimental conditions. As an example, for 0.01 rad. length, momentum 1.5 GeV/c, one finds $\Delta\alpha \simeq$ 1 mrad. With L_o=1m , n_o=25, and σ=0.5 mm, ρ=10 ; the condition $\rho>1$ is then widely satisfied.

If $\rho \gg 1$ both minima of V_{ZZ}^t and V_{bb}^t occur for $\frac{n-n_o}{n_o} \simeq \rho^{-2/3}$ i.e. for n barely greater than n_o. Then the track is probably measured beyond this minimum.

This leads us to the notion of optimal measurement length (eventually less than the available length), already used for bubble chambers : using points beyond this length worsens the precision of the standard estimators. Unfortunately the optimal length is not, in general, the same for Z and b.

Fig. 1 represents $\sqrt{V^t_{ZZ}}$ and $\sqrt{V^t_{bb}}$ as functions of $\frac{n}{n_o}$ for several values of ρ .

Let us now evaluate the scattering contribution to $\langle\chi^2\rangle$:

$$\sigma^2 \, (\chi^2)^S = \sum_{i\leqslant n_o} (Z + bx_i)^2 + \sum_{i>n_o} \left[Z + bx_i - \alpha\ell(i-n_o)\right]^2$$

Knowing the expressions of $\langle Z^2\rangle$,$\langle b^2\rangle$,$\langle Zb\rangle$,$\langle\alpha^2\rangle$, $\langle\alpha Z\rangle$ and $\langle\alpha b\rangle$ from (2.4), one finds $\langle\chi^2\rangle^S$:

$$\langle\chi^2\rangle^S = \frac{n_o^3(n-n_o)^3}{3n^3} \frac{\ell^2\Delta\alpha^2}{\sigma^2} \qquad (2.6)$$

Hence $$\frac{\langle\chi^2\rangle^S}{n} = \frac{n_o(n-n_o)^3}{3n^4} \frac{L_o^2\Delta\alpha^2}{\sigma^2} = \frac{(n-n_o)^3}{3n^4} \rho^2 \qquad (2.7)$$

For fixed n_o, $\frac{\langle\chi^2\rangle^S}{n}$ rises to a maximum (at $n = 4n_o$) :

$$\frac{9}{256} \frac{\rho^2}{n_o} = 0.0352 \frac{\rho^2}{n_o}$$

If $\rho \gg 1$, $\frac{\langle\chi^2\rangle^S}{n} \simeq \frac{1}{3n_o}$ with the optimal length.

With the same numerical values as above, one finds $\frac{\langle\chi^2\rangle^S}{n} < .14$ while $\frac{\langle\chi^2\rangle^m}{n} \simeq 1$. In other terms $\frac{\chi^2}{n}$ remains in mean value close to 1 (although its distribution may differ from the standard one). Generally the multiple scattering introduces a relative increase much smaller for the $\langle\chi^2\rangle$ than for the variances of the fitted parameters.

2.2.3. Parabolic fit :

The calculations are more complicated, but quite analogous. The equation of the trajectory is :

$$y = Y + a\,x + \frac{c}{2}\,x^2$$

The parameters Y,a and c are deduced from :

$$\begin{bmatrix} n & \sum_i x_i & \sum_i x_i^2 \\ \cdots & \sum_i x_i^2 & \sum_i x_i^3 \\ \cdots & \cdots & \sum_i x_i^4 \end{bmatrix} \begin{bmatrix} Y \\ a \\ \frac{c}{2} \end{bmatrix} = \begin{bmatrix} \sum_i y_i^m \\ \sum_i x_i\, y_i^m \\ \sum_i x_i^2 y_i^m \end{bmatrix}$$

(from now on, for symmetric matrices, we write explicitly the upper triangle only).

Hence the measurement contribution to the covariance matrix of Y, a and $\frac{c}{2}$:

$$V^m \simeq \sigma^2 \begin{bmatrix} n & \frac{n^2}{2}\ell & \frac{n^3}{3}\ell^2 \\ \cdots & \frac{n^3}{3}\ell^2 & \frac{n^4}{4}\ell^3 \\ \cdots & \cdots & \frac{n^5}{5}\ell^4 \end{bmatrix}^{-1} = \frac{\sigma^2}{n} \begin{bmatrix} 9 & -\frac{36}{L} & \frac{30}{L^2} \\ \cdots & \frac{192}{L^2} & -\frac{180}{L^3} \\ \cdots & \cdots & \frac{180}{L^4} \end{bmatrix} \qquad (2.8)$$

Taking into account the scattering only :

$$\sum_i y_i^m = \frac{(n-n_o)^2}{2}\,\ell\,\alpha$$

$$\sum_i x_i\, y_i^m = \frac{(n-n_o)^2(2n+n_o)}{6}\,\ell^2\alpha$$

$$\sum_i x_i^2 y_i^m = \frac{(n-n_o)^2(3n^2+2n_o n+n_o^2)}{12}\,\ell^3\alpha$$

Hence the scattering contribution V^s :

$$V^s = (n-n_o)^4 \Delta\alpha^2 \times$$

$$\begin{bmatrix} \frac{n_o^2(2n-5n_o)^2}{4n^6}\ell^2 & \frac{n_o(5n_o-2n)(n-3n_o)(n+5n_o)}{2n^7}\ell & \frac{15n_o^3(5n_o-2n)}{n^8} \\ \cdots & \frac{(n-3n_o)^2(n+5n_o)^2}{n^8} & \frac{30n_o^2(n-3n_o)(n+5n_o)}{n^9\ell} \\ \cdots & \cdots & \frac{225n_o^4}{n^{10}\ell^2} \end{bmatrix}$$

(2.9)

Here again V^t_{YY} and V^t_{aa} tend to foreseeable finite limits when $n\to\infty$ ($L_o^2\Delta\alpha^2$ and $\Delta\alpha^2$ respectively; the same as V^t_{ZZ} and V^t_{bb} in the straight fit) while V^t_{cc} goes to zero. Thus the uncertainty of any kinematical quantity which is not a function of the curvature only, does not vanish for $n \to \infty$.

Studying the variance of Y, a and c as functions of n leads to solve algebraic equations ; in table 1 we give the variations of these variances, according to the value of the same parameter $\rho = \frac{\sqrt{n_o}L_o\Delta\alpha}{\sigma}$ as above (2.5).

This table shows the existence, in many cases, of an optimal length ; again it depends on the parameter for which the best precision is required. The optimal length lies in general either just after L_o , or much higher.

Fig. 2 represents $\sqrt{V^t_{YY}}$, $\sqrt{V^t_{aa}}$ and $\sqrt{V^t_{cc}}$ as functions of n/n_o ; there functions exhibit a remarkable feature : since V^s_{YY} vanished for $n = \frac{5}{2} n_o$ and V^s_{aa} for $n = 3n_o$, the curves of $\sqrt{V^t_{YY}}$ and $\sqrt{V^t_{aa}}$ are tangent at these points, for any ρ , to the straight lines (in log-log coordinates) obtained without scattering ($\rho = 0$).

The values of ρ chosen for fig. 2 extend higher than those for fig. 1, because the value of σ is generally smaller perpendicular to the field than parallel to it, while $\Delta\alpha$, of course, is the same.

For $\langle\chi^2\rangle$, the same method as in 2.2.2 leads to :

$$\frac{\langle\chi^2\rangle^S}{n} = \frac{(n-n_o)^3(4n^2-15n_on+15n_o^2)}{12\ n^6}\ \rho^2 \qquad (2.10)$$

As a function of n , this quantity reaches, for $n = 8.162\ n_o$, a maximum equal to $0.01647\ \frac{\rho^2}{n_o}$ and remains less than $0.01\ \frac{\rho^2}{n_o}$ for $n < 4n_o$.

Thus the extra degree of freedom brought by the curvature leads to a smaller χ^2 relative increase than in the straight line fit.

If $\rho \gg 1$, $\frac{\langle\chi^2\rangle^S}{n} \simeq \frac{1}{3n_o}$ for the length corresponding to the first minimum of V^t_{YY} , V^t_{aa} and V^t_{cc} (which is not necessarily the optimal length).

2.3. Scattering uniformily distributed along the track :

2.3.1. Principle of the calculation :

Each matter slice between two measured points causes random scattering. Since the scattering angles are independent, their contributions to the variances and to $\langle\chi^2\rangle$ are additive : so any configuration of matter can be handled starting from the results found in 2.2. We will here study the case of uniformly distributed matter (like Drijard (2)) : this is exact for bubble chambers and realistic for many other detectors.

The covariance matrix and the $\langle\chi^2\rangle$ will be obtained, for a given value of n, by integration over n_o, considered as a con-

tinuous variable . $\delta\alpha^2$ is the mean squared scattering angle over a length ℓ.

2.3.2. Straight line fit :

From (2.4) we get :

$$V^s_{ZZ} = \frac{\ell^2\delta\alpha^2}{n^4}\int_o^n dx\, x^2(n-x)^4 = \frac{1}{105}\, n^3\, \ell^2\delta\alpha^2$$

$$V^s_{bb} = \frac{\delta\alpha^2}{n^6}\int_o^n dx\, (2x+n)^2(n-x)^4 = \frac{13}{35}\, n\, \delta\alpha^2 \qquad (2.11)$$

$$V^s_{Zb} = -\frac{\ell\delta\alpha^2}{n^5}\int_o^n dx\, x(2x+n)(n-x)^4 = -\frac{11}{210}\, n^2\, \ell\, \delta\alpha^2$$

If ℓ is fixed, the terms of V^s increase with n (as n^k) while the corresponding terms of V^m , given by (2.2), decrease (as n^{k-4}) : then the total variances have a minimum. One finds, with $q = \frac{\ell\,\delta\alpha}{\sigma}$:

$$\text{For } V^t_{ZZ} : \quad n_{min} \simeq \frac{3.44}{\sqrt{q}} \quad ; \quad (V^t_{ZZ})_{min} = \frac{4}{3}\, V^m_{ZZ}(n_{min})$$

$$\text{For } V^t_{bb} : \quad n'_{min} \simeq \frac{3.14}{\sqrt{q}} \quad ; \quad (V^t_{bb})_{min} = 4\, V^m_{bb}(n'_{min}) \qquad (2.12)$$

Fig. 3 shows the variation of $\sqrt{V^t_{ZZ}}$ and $\sqrt{V^t_{bb}}$; the shape of the curves is independent of the values of ℓ , σ , $\delta\alpha$.

With plausible numerical values (momentum 1.5 GeV/c, ℓ = 5 cm , σ = 0.5 mm, rad. length = 100m)one finds $\frac{1}{\sqrt{q}} = 6.7$, whence the optimal lengths : 1.15 m for Z and 1.05 m for b . So the measurements may often exceed the optimum.

$|V^t_{Zb}|$ has also a minimum, for $n \simeq 3.27/\sqrt{q}$ .Hence for any function of Z and b , the optimum lies close to $3.25/\sqrt{q}$.

From (2.6) the calculation of $\langle\chi^2\rangle$ gives straightly :

$$\langle\chi^2\rangle^s = \frac{\ell^2\delta\alpha^2}{3\sigma^2}\int_o^n dx\, \frac{x^3(n-x)^3}{n^3} = \frac{n^4\, q^2}{420}$$

$$\frac{\langle\chi^2\rangle^s}{n} = \frac{n^3\, q^2}{420} \qquad (2.13)$$

In pratical conditions the relative increase of $\langle\chi^2\rangle$ due to the scattering is negligible compared to that of the variances. When $n \to \infty$ it behaves as n^3 instead of n^4 for the variances . For the optimal length ($n \simeq 3.25/\sqrt{q}$), $\langle\chi^2\rangle^s \simeq 0.266$ while $\langle\chi^2\rangle^m \simeq n$:so the mean squared residual remains nearly the same as without scattering.

2.3.3. Parabolic fit :

For the parameters Y,a,$\frac{c}{2}$ one obtains :

$$V^s = \sigma_x^2 \begin{bmatrix} \frac{n^3 x^2}{630} & -\frac{n^2 x}{60} & \frac{n}{168} \\ \cdots & \frac{8n}{35} & -\frac{3}{28 x} \\ \cdots & \cdots & \frac{5}{14 n^3 x^2} \end{bmatrix} \qquad (2.14)$$

Here again the dependance on n is n^k while the corresponding terms of V^m behave as n^{k-4} .

V_{YY} , V_{aa} and V_{Ya} behave like V_{ZZ} , V_{bb} and V_{Zb} in the straight fit. The optimal lengths are respectively $6.59/\sqrt{q}$, $7.09/\sqrt{q}$ and $6.82/\sqrt{q}$, i.e. about twice their values for Z and b, if the precision is the same in both directions (as already mentioned, many detectors have higher precision in the curvature plane, and the order of the optimal lengths may be reversed).

The c parameter appears to be quite different : V^s_{cc} is a decreasing function of n, and there is no optimal length for the curvature. When its uncertainty has the highest weight in the kinematical analysis, the track should be measured over the greatest possible length. In other configurations it may be preferable to truncate some tracks.

This is true also for geometrical questions related to the position uncertainty (e.g. attaching a track to a vertex).

The satisfactory solution would be to perform several fits on each track, in order to optimize separately the parameters : but it would increase substantially the computation time, and the covariance non-diagonal terms could not be extracted any more from an already calculated matrix.

$\langle\chi^2\rangle$ can be evaluated in the same way as in 2.3.2. :

$$\langle\chi^2\rangle^s = \frac{\ell^2\delta\alpha^2}{n^5\sigma^2} \int_o^n dx \frac{x^3(n-x)^3(4n^2-15nx+15x^2)}{12} = \frac{n^4q^2}{2520}$$

$$\text{Hence} \quad \frac{\langle\chi^2\rangle^s}{n} = \frac{n^3q^2}{2520} \tag{2.15}$$

The numerical coefficient is smaller than in (2.13) because of the extra degree of freedom. For the optimal length associated to Y and a ($n \simeq 6.85/\sqrt{q}$), $\langle\chi^2\rangle^s = 0.874$, negligible compared to $\langle\chi^2\rangle^m \simeq n$

3. THE OPTIMAL FIT : RECURSIVE METHOD

3.1. Description of the method :

3.1.1. General features :

The track is measured at abscissas $x_1, x_2 \ldots x_N$. Knowing the best estimators of the track parameters at x_n, starting from the measurements at x_n, $x_{n+1} \ldots x_N$, we want to define the best estimators of the parameters at x_{n-1} starting from the measurements at x_{n-1}, x_n , ... x_N . Here we have to do with the true parameters, i.e. including the effect of the previous scatterings.

This involves :

- scattering between x_{n-1} and x_n, that is, loss of information w.r.t the simple extrapolation of the parameters and of their covariance matrix.

- extra information thanks to the x_{n-1} measurement.

We suppose the distance δx_n between x_{n-1} and x_n to be short, so that the scattering affects the angles but not the posi-

tion : in other terms we can consider that the matter is concentrated at the measurement points, and that the track can be assimilated to a broken line ; anyway, one can introduce intermediate scattering points where no measurement is made. This assumption simplifies the calculations, but it is not essential to the method.

We will first describe this (backward) recursive estimator for tracks in a plane, parametrizing each elementary segment as a portion of parabola (with the same notations as in sect. 2).

3.1.2. Accounting for the scattering :

Let y_n^t , a_n^t and c_n^t be the true values of the parameters at x_n, after scattering ; the true values at this point before scattering are y_n^t, $a_n^t+\alpha_n$ and c_n^t , where α_n is a quasi-gaussian variable of variance $\delta\alpha_n^2$. Let y_n^{opt} , a_n^{opt} and c_n^{opt} be the best estimators of y_n^t , a_n^t and c_n^t in terms of y_n^m , $y_{n+1}^m \ldots y_N^m$ (measured coordinates at x_n, $x_{n+1} \ldots x_N$) and V_n their covariance matrix. These estimators, as random variables, are clearly independent of α_n . Then they are also the best estimators of y_n^t , $a_n^t+\alpha_n$ and c_n^t , with a covariance matrix V_n^* equal to V_n where $\delta\alpha_n^2$ has been added to the V_{aa} term.

In the framework of information theory : if a_n^{opt} is an efficient estimator of a_n^t , it is also efficient for $a_n^t+\alpha_n$.

3.1.3. Propagating to the previous point :

The transformation from the parameters at x_n (before scattering) to those at x_{n-1} (after scattering) is one-to-one, without information loss. Then y'_{n-1}, a'_{n-1} and c'_{n-1} defined by :

$$\begin{bmatrix} y'_{n-1} \\ a'_{n-1} \\ c'_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -\delta x_n & \frac{\delta x_n^2}{2} \\ 0 & 1 & -\delta x_n \\ 0 & 0 & 1 \end{bmatrix}}_{D_n} \begin{bmatrix} y_n^{opt} \\ a_n^{opt} \\ c_n^{opt} \end{bmatrix} \qquad (3.1)$$

are the best estimators of y^t_{n-1}, a^t_{n-1} and c^t_{n-1} in terms of y^m_n , y^m_{n+1}, ... y^m_N . Their covariance matrix is :

$$V'_{n-1} = D_n \left(V_n + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta\alpha_n^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) D_n^t = D_n V_n^* D_n^t \qquad (3.2)$$

3.1.4. Adding the previous measurement :

Now we add the information brought by y^m_{n-1} . This information concerns y^t_{n-1}, a^t_{n-1} and c^t_{n-1} , and also α_n , which is not useful for our purpose (it could perhaps be used for particle identification at low momentum).

The likelihood of the whole set of measurements (y^m_n ... y^m_N), as a function of y, a , c (parameters to be fitted at x_{n-1}) is given, up to a constant factor, by exp $-\frac{1}{2}\,\Delta p^t (V'_{n-1})^{-1}\,\Delta p$ where the components of Δp are $y-y'_{n-1}$, $a-a'_{n-1}$ and $c-c'_{n-1}$. The likelihood of the y^m_{n-1} measurement is simply exp $-(y-y^m_{n-1})^2/2\sigma^2$.

Since the errors are independent, the likelihood functions can be multiplied ; the maximum likelihood is obtained by minimizing the quadratic function :

$$\Delta p^t \; (V'_{n-1})^{-1} \; \Delta p \; + \; \frac{(y-y^m_{n-1})^2}{\sigma^2} \qquad (3.3)$$

with respect to y, a , c .

The best estimators y^{opt}_{n-1} , a^{opt}_{n-1} and c^{opt}_{n-1} are then solution of the linear system :

$$\left((V'_{n-1})^{-1} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y-y'_{n-1} \\ a-a'_{n-1} \\ c-c'_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{y^m_{n-1}-y'_{n-1}}{\sigma^2} \\ 0 \\ 0 \end{bmatrix} \qquad (3.4)$$

and their covariance matrix :

$$V_{n-1} = \left((V'_{n-1})^{-1} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \qquad (3.5)$$

Equations (3.2) to (3.5) can be written with the information matrices (i.e. the inverted covariance matrices, in the gaussian hypothesis)

(3.2) gives :

$$I'_{n-1} = (D_n^t)^{-1} \underbrace{\left((I_n)^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta\alpha_n^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1}}_{I_n^*} (D_n)^{-1}$$

with

$$(D_n)^{-1} = \begin{bmatrix} 1 & \delta x_n & \frac{\delta x_n^2}{2} \\ 0 & 1 & \delta x_n \\ 0 & 0 & 1 \end{bmatrix}$$

Moreover :

$$I_n^* = \left((I_n)^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta\alpha_n^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} = I_n - \frac{\delta\alpha_n^2}{1+\delta\alpha_n^2 (I_n)_{aa}} I_n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} I_n$$

$$= I_n - \underbrace{\frac{\delta\alpha_n^2}{1+\delta\alpha_n^2 (I_n)_{aa}} \begin{bmatrix} (I_n)_{Ya}^2 & (I_n)_{Ya}(I_n)_{aa} & (I_n)_{Ya}(I_n)_{Yc} \\ \cdots & (I_n)_{aa}^2 & (I_n)_{aa}(I_n)_{ac} \\ \cdots & \cdots & (I_n)_{ac}^2 \end{bmatrix}}_{\text{information loss}} \qquad (3.6)$$

Equation (3.4) becomes :

$$\left(\underbrace{(D_n^t)^{-1} \; I_n \; (D_n)^{-1}}_{I'_{n-1}} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y-y'_{n-1} \\ a-a'_{n-1} \\ c-c'_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{y^m_{n-1}-y'_{n-1}}{\sigma^2} \\ 0 \\ 0 \end{bmatrix} \qquad (3.7)$$

and (3.5) gives :

$$I_{n-1} = I'_{n-1} + \underbrace{\begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{information brought by the measurement}} \qquad (3.8)$$

This formulation avoids the explicit matrix inversions involved in (3.4) and (3.5).

Table 2 summarizes the recursive algorithm.

3.1.5. Starting the recursion :

This can be done by two practically equivalent means :

- Using the last three points to calculate y, a , c at x_{N-2} , and their covariance matrix (including the scattering contribution at x_{N-2} and x_{N-1} if necessary).

- Starting the recursion at the last point, which defines y_N with variance σ^2 , but lets a and c fully undetermined : one can assign to them approximate initial values and very large arbitrary variances : then the recursive algorithm can be applied without risk of matrix singularity : the information brought by y^m_{N-2} fits a and c at x_{N-2}, and the fitted values are nearly independent of the initial ones.

3.2. Generalizations :

We consider now a track in 3-dimensional space with an inhomogeneous magnetic field : it is defined by 5 parameters (e.g. y, z, a, b

and 1/p at fixed x). Moreover the measured coordinates are not necessarly y or z at fixed x.

We describe hereafter the 3 stages of each step of the recursion. Instead of the azimuth ϕ and the dip λ , we continue to use the slope parameters $a=p_y/p_x=\tan\phi$ and $b=p_z/p_x= \tan\lambda/\cos\phi$: this avoids trigonometrical calculations. We use also : $d = 2e/p$ (B.d is then the curvature for a track perpendicular to the magnetic field).

3.2.1. Scattering at x_n :

It affects only λ and ϕ (or a and b). Let ξ_n be the number of radiation lengths X_o crossed from x_{n-1} to x_n. For a homogeneous medium :

$$\xi_n = \frac{x_n - x_{n-1}}{X_o \cos\lambda \cos\phi} = \frac{\delta x_n}{X_o} \sqrt{1+a^2+b^2} \qquad (3.9)$$

The scattering angles α and β in two perpendicular planes containing the track direction are independent random variables of variance $\delta\alpha_n^2 = (\frac{KZ}{pv})\xi_n$. If one of these planes is vertical, the scattering angle in this plane represents exactly the variation of λ and the other one the variation of ϕ multiplied by $\cos\lambda$; with the parameters ϕ and λ to account for the scattering consists in adding $\delta\alpha_n^2/\cos^2\lambda$ to $V_{\phi\phi}$ and $\delta\alpha_n^2$ to $V_{\lambda\lambda}$.

With the parameters a and b, one finds easily, that one has to add :

$$\begin{cases} (1+a^2+b^2)(1+a^2)\ \delta\alpha_n^2 & \text{to} \quad V_{aa} \\ (1+a^2+b^2)(1+b^2)\ \delta\alpha_n^2 & \text{to} \quad V_{bb} \\ (1+a^2+b^2)\ ab\ \ \delta\alpha_n^2 & \text{to} \quad V_{ab} \end{cases} \qquad (3.10)$$

In terms of information matrices, we get :

$$I_n^* = (V_n^*)^{-1} = \left((I_n)^{-1} + \delta\alpha_n^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} & 0 \\ 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)^{-1} \qquad (3.11)$$

where $A = (1 + a^2 + b^2) \begin{bmatrix} 1 + a^2 & ab \\ ab & 1 + b^2 \end{bmatrix}$

The matrix inversions can be simplified in the same way as in 3.1.4. :

$$I_n^* = I_n - \delta\alpha_n^2 \, I_n \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} I_n \qquad (3.12)$$

with $$a = \left(A^{-1} + \delta\alpha_n^2 \begin{bmatrix} I_{aa} & I_{ab} \\ I_{ab} & I_{bb} \end{bmatrix} \right)^{-1}$$

3.2.2. Propagation from x_n to x_{n-1} :

The track extrapolation from a point to the previous one must be accurate : the cumulated error over the whole length of the track should remain small with respect to the position uncertainty resulting from the fit. Any suitable method for this extrapolation allows to calculate also the (5x5) matrix of derivatives analogous to D_n in (3.1), either by analytical differentiation, or by finite differences.

As a matter of fact, in the left-hand side of the equation which should generalize (3.4), the matrix V'_{n-1} can be calculated with an approximation of D_n , assuming the field B to be constant along the track between x_{n-1} and x_n. If the curvature is not too strong, we obtain for the parameters y, z, a, b and d :

$$D_n \simeq \begin{bmatrix} 1 & 0 & -\delta x_n + F_1 \delta x_n^2/2 & F_2 \delta x_n^2/2 & F_3 \delta x_n^2/2 \\ 0 & 1 & G_1 \delta x_n^2/2 & -\delta x_n + G_2 \delta x_n^2/2 & G_3 \delta x_n^2/2 \\ 0 & 0 & 1 - F_1 \delta x_n & -F_2 \delta x_n & -F_3 \delta x_n \\ 0 & 0 & -G_1 \delta x_n & 1 - G_2 \delta x_n & -G_3 \delta x_n \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.13)$$

with : $F_3 = e\left[bB_x + abB_y - (1+a^2)B_z\right]$; $F_1 = d\left[aF_3/e^2 + e(bB_y - 2aB_z)\right]$

$F_2 = d\left[bF_3/e^2 + e(B_x + aB_y)\right]$

$G_3 = e\left[-aB_x + (1+b^2)B_y - abB_z\right]$; $G_1 = d\left[aG_3/e^2 - e(B_x - bB_z)\right]$

$G_2 = d\left[bG_3/e^2 + e(2bB_y - aB_z)\right]$

where $e^2 = 1 + a^2 + b^2$

3.2.3. Addition of a measurement :

In many cases each measurement consists in determining y or z, or a combination $t=\lambda y+\mu z$, at a given fixed x. Here it is useless to build space points from several raw measurements : such a procedure implies sometimes a loss of information.

The likehood of measurement t^m_{n-1} is now, in the gaussian approximation :

$$\exp \quad -\frac{1}{2} \quad \frac{(\lambda y+\mu z-t^m_{n-1})^2}{\sigma^2_n}$$

and the linear system (3.4) becomes :

$$\left((V'_{n-1})^{-1} + \frac{1}{\sigma^2} \begin{bmatrix} \lambda^2 & \lambda\mu & 0 & 0 & 0 \\ \lambda\mu & \mu^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y-y'_{n-1} \\ z-z'_{n-1} \\ a-a'_{n-1} \\ b-b'_{n-1} \\ d-d'_{n-1} \end{bmatrix} = \frac{1}{\sigma^2} \begin{bmatrix} \lambda(t^m_{n-1}-\lambda y'_{n-1}-\mu z'_{n-1}) \\ \mu(t^m_{n-1}-\lambda y'_{n-1}-\mu z'_{n-1}) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.14}$$

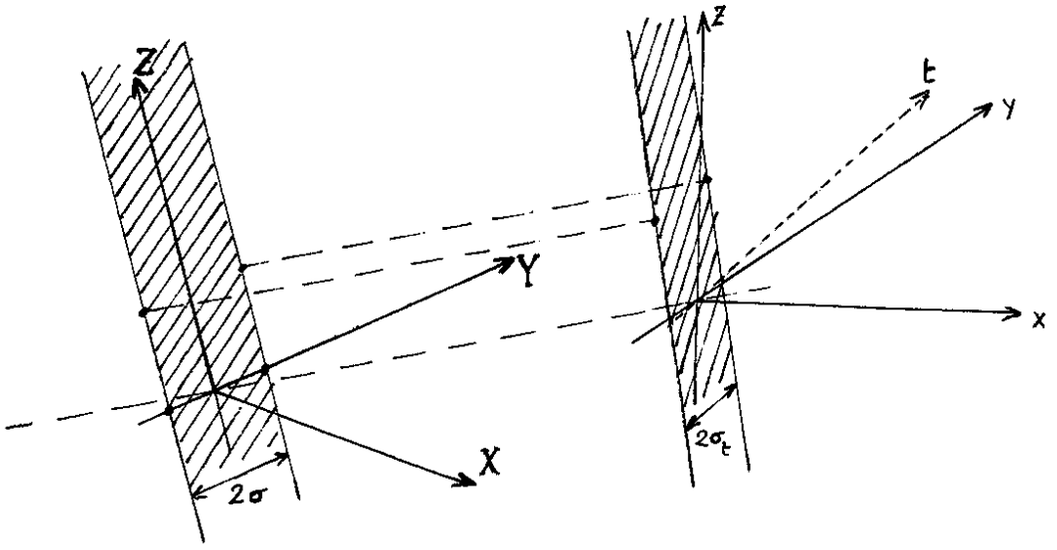
Of course, if y and z (or two independent combinations of y and z) ara available at the same x, both measurements can be included together in one equation like (3.14).

If the information refers to a non linear combination of y and z, it can generally be linearized around the measured values.

If the measured quantity depends also on x, especially in the case of optical measurements, one can transform the raw information into information at fixed x : this transformation is a projection along the direction of the track : so it requires a first approximation of the track to determine an approximate abscissa of the measured point and its direction at this point.

As an example, let us suppose we measure Y at fixed X, with error σ , in an orthogonal frame XYZ defined in the lab. by the unit vectors (u_x, u_y, u_z) for X direction, (v_x, v_y, v_z) for Y direction and (w_x, w_y, w_z) for Z direction.

This can be expressed by assigning error σ to Y and infinity to Z, and represented by a "standard uncertainty strip".



(the origins of the frames are separated for clarity ; in fact they coincide).

To express the error matrix in (y,z) variables at fixed x amounts to project the points of the (YZ) plane onto the (y z) plane along the track direction, i.e. the vector (1, a, b) : in the gaussian hypothesis the projection of a standard strip is also a standard strip.

This projection is expressed by :

$$\begin{cases} y = v_y\, Y + w_y\, Z - a(v_x Y + w_x Z) = (v_y - a v_x)\, Y + (w_y - a w_x) Z \\ z = v_z\, Y + w_y\, Z - b(v_x Y + w_x Z) = (v_z - b v_x)\, Y + (w_z - b w_x) Z \end{cases}$$

The direction of the projected strip id defined by the projection of the Z direction, i.e. $(w_y - a w_x, w_z - b w_x)$ in the (yz) plane. In this plane, the variable t perpendicular to the strip is (up to an arbitrary factor) :

$$t = (w_z - b w_x) y - (w_y - a w_x) z \text{ , i.e. :}$$

$$t = (v_y - a v_x)(w_z - b w_x) - (v_z - b v_x)(w_y - a w_x)\, Y$$

$$\text{or } t = (u_x + a u_y + b u_z)\, Y \qquad (3.15)$$

whence the variance of t :

$$\sigma_t^2 = (u_x + a u_y + b u_z)^2\, \sigma^2 \qquad (3.16)$$

Then equation (3.14) can be used.

An optical measurement with a camera (as in bubble chambers) consists in projecting some points of the track onto a plane, from a fixed sighting point. It can be handled in a similar way : indeed, even if the projection is not orthogonal, one can define locally a frame XYZ in which the error matrix is diagonal and σ_Z infinite (Z is the sighting direction). Let α, β , γ be the direction cosines of the track in this frame. From the point of view of the information on the track, the elliptical uncertainty cylinder in the Z direction, with half-axes σ_X and σ_Y , is equivalent to an uncertainty strip in any plane containing the Z axis.

By choosing the (YZ) plane, the projection Y', Z' of X, Y, Z onto this plane according to the direction (α, β, γ) reads :

$$\begin{cases} Y' = Y - \dfrac{\beta}{\alpha} X \\ Z' = Z - \dfrac{\gamma}{\alpha} X \end{cases} \qquad (3.17)$$

Hence the variance of Y' :

$$\sigma_{Y'}^2 = \sigma_Y^2 + \left(\frac{\beta}{\alpha}\right)^2 \sigma_X^2 \qquad (3.18)$$

So we are led back to the previous problem.

When several measurements are made close together (e.g. in successive planes of a MWPC), they can be included in the same step of the recursion. For example, let us assume that beside the measurement at x, the coordinate y is measured with variance σ^2 at $x + \Delta x$ (Δx small). Let y', z', a', b', d' be the values of the parameters at x,

fitted from the down stream measurements.

In order to calculate the contribution of the measurement y^m at $x + \Delta x$ to the log. likelihood of the set of parameters (y, z, a, b, d) at x, we express $y(x + \Delta x)$ as a function of these parameters :

$$y\ (x + \Delta x) = Y\ (y,z,a,b,d)$$

and then we write :

$$Y(y,z,a,b,d) = Y(y',z',a',b',d') + \sum \frac{\partial Y}{\partial p_i}\ (p_i - p_i')$$

where $\{p_i\} = y,z,a,b,d$

Hence the log. likelihood :

$$-\frac{[y(x+\Delta x)-y^m]^2}{2\sigma^2} = -\frac{\left[Y(y',z',a',b',d')-y^m + \sum_i \frac{\partial Y}{\partial p_i}\ (p_i-p_i')\right]^2}{2\sigma^2}$$

Thus accounting for this measurement leads to add, in eq. (3.14) :

$\frac{1}{\sigma^2}\ \frac{\partial Y}{\partial p_i}\ \frac{\partial Y}{\partial p_j}$ to the ij term of the left hand side matrix

$\frac{1}{\sigma^2}\ \frac{\partial Y}{\partial p_i}\ [y^m - Y(y',z',a',b',d')]$ to the i term of the right hand side vector.

Since Δx is small, we need merely an approximate expression for Y, e.g. :

$$Y(y,z,a,b,d) = y + a\ \Delta x + d\ \sqrt{1+a^2+b^2}\ [bB_x + abB_y - (1+a^2)B_z]\ \frac{\Delta x^2}{2}$$

The derivatives $\frac{\partial Y}{\partial p_i}$ can even be reduced to their simplest expressions :

$$\frac{\partial Y}{\partial y} = 1\ ;\ \frac{\partial Y}{\partial a} = \Delta x\ ;\ \frac{\partial Y}{\partial z} = \frac{\partial Y}{\partial b} = \frac{\partial Y}{\partial d} = 0$$

This treatment extends straightforward to several measurements of any combination of y and z in the neighbourhood of x.

3.3. Extension to energy loss :

The energy loss can be taken into account in a natural way : it is enough, in the propagation stage 3.2.2. , to add a suitable quantity to the d parameter. Moreover, it is possible to introduce a (gaussian) uncertainty on this energy loss, as in 3.2.1., by adding a contribution to V_{dd} in the same time as V_{aa}, V_{bb} and V_{ab} at the scattering stage.

4. COMPARISON BETWEEN THE OPTIMAL FIT AND THE STANDARD FIT :

4.1. Introduction :

Our aim is to determine the gain in precision on the parameters, brought by the optimal estimators, with the same model as in 2.2. and 2.3., and the same notations.

In the optimal fit, the variances of the parameters cannot increase as measurement points are added (in other terms, addition of measurement cannot cause a loss of information). Thus, for $n \to \infty$ they must tend towards some limits (possibly zero). Again we use the gaussian approximation : so the covariance matrix V of the fitted parameters is the inverse of the information matrix I.

4.2. Scattering at one point :

4.2.1. Principle of the calculation :

The expressions of the measured variables (2.1) show that the measurements after L_o are independent of those before L_o, so that the total information matrix on the parameters (Z and b) at the

origin is the sum of the contributions brought by the two portions of track delimited by the scatterer

For the second portion (after L_o), we calculate first the information on the parameters at $x = L_o$ (including the effect of the scattering uncertainty), and then we propagate it back to $x = 0$. At $x = 0$ we add the information of the first portion (before L_o), where no scattering uncertainty occurs.

4.2.2. Straight line fit :

For the parameters at $x = L_o$ after scattering, the second portion provides an information matrix which can be calculated with the standard estimator defined by (2.2) :

$$I_2(L_o) = \frac{1}{\sigma^2} \begin{bmatrix} \Delta n & \frac{\Delta n^2}{2}\ell \\ \dots & \frac{\Delta n^3}{3}\ell^2 \end{bmatrix} \quad \text{with} \quad \Delta n = n - n_o$$

Hence the information matrix on the parameters at $x = L_o$ before scattering, following the arguments developed in 3.1.2. :

$$I_2^*(L_o) = \left((I_2(L_o))^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & \Delta\alpha^2 \end{bmatrix} \right)^{-1}$$

$$= \frac{\Delta n}{\sigma^2} \frac{1}{4(3+\zeta)} \begin{bmatrix} 12+\zeta & 6\Delta n\ell \\ \dots & 4\Delta n^2\ell^2 \end{bmatrix}$$

With $$\zeta = \frac{\Delta n^3 \ell^2 \Delta\alpha^2}{\sigma^2}$$

Propagating this information to $x = 0$ gives :

$$I_2'(0) = \begin{bmatrix} 1 & 0 \\ L_o & 1 \end{bmatrix} I_2^*(L_o) \begin{bmatrix} 1 & L_o \\ 0 & 1 \end{bmatrix}$$

$$= \frac{\Delta n}{\sigma^2} \frac{1}{4(3+\zeta)} \begin{bmatrix} 12+\zeta & \left[(12+\zeta)n_o + 6\Delta n\right]\ell \\ \dots & \left[(12+\zeta)n_o^2 + 12n_o\Delta n + 4\Delta n^2\right]\ell^2 \end{bmatrix} \qquad (4.1)$$

The first portion gives straightly an information on Z and b at x = 0 :

$$I_1(0) = \frac{1}{\sigma^2}\begin{bmatrix} n_o & \frac{n_o^2\ell}{2} \\ \cdots & \frac{n_o^3\ell^2}{3} \end{bmatrix}$$

Hence the total information matrix on Z and b :

$$I_o = I_1(0) + I_2'(0) = \frac{n_o}{4\sigma^2(3+\zeta)}\begin{bmatrix} 3\zeta(4\eta+1)+12\eta & \zeta(\eta+1)+6\eta^2 \\ \cdots & \zeta(\eta+\frac{1}{3})+4\eta^3 \end{bmatrix} \qquad (4.2)$$

with : $\eta = \frac{n}{n_o}$

I_o^{-1} gives the covariance matrix of the optimal estimators.

The optimal errors on Z and b are represented on fig. 1 together with the standard ones, in order to show the gain in precision.

We will calculate explicitly their limits for $n \to \infty$ (i.e. $\eta \to \infty$) with n_o fixed. Remarking that $\zeta = \rho^2\eta^3$ (ρ^2 was already defined by (2.5)), we keep the first two terms in powers of η :

$$I_o \simeq \frac{n_o}{4\sigma^2}\begin{bmatrix} \eta & (\eta+1)n_o\ell \\ \cdots & (\eta+\frac{1}{3}+\frac{4}{\rho^2})\, n_o^2\ell^2 \end{bmatrix} \qquad (4.3)$$

With the terms in η only, I_o would be singular : this reflects the fact that the combination $Z + bL_o$ is known, from the second portion, with an infinite precision, whereas the precisison is limited by the scattering for any other combination.

At first order in $1/\eta$, the covariance matrix is :

$$V_o = (I_o)^{-1} \simeq \frac{\sigma^2}{n_o(\frac{1}{\rho^2}+\frac{1}{3})}\begin{bmatrix} 1 & -\frac{1}{L_o} \\ \cdots & \frac{1}{L_o^2} \end{bmatrix} \qquad (4.4)$$

whence the limits for $n \to \infty$:

$$V_{ZZ}^{\infty} = L_o^2 \; V_{bb}^{\infty} = \frac{\sigma^2}{n_o(\frac{1}{\rho^2} + \frac{1}{3})} \qquad (4.5)$$

This quantity is always less than $3\sigma^2/n_o$.

Generally V_{bb}^{∞} is small compared to the minimum of the standard variance (cf. 2.2.2.) ; however, as it can be seen on fig. 1, the convergence of $(V_n)_{bb}$ to its limit is very slow, and the gain with respect to the standard fit applied up to the optimal length, remains often moderate.

4.2.3. Parabolic fit :

With the same notations as in 4.2.2., we have now for the parameters Y, a, and $\frac{c}{2}$ after scattering :

$$I_2(L_o) = \frac{1}{\sigma^2} \begin{bmatrix} \Delta n & \frac{\Delta n^2}{2}\ell & \frac{\Delta n^3}{3}\ell^2 \\ \dots & \frac{\Delta n^3}{3}\ell^2 & \frac{\Delta n^4}{4}\ell^3 \\ \dots & \dots & \frac{\Delta n^5}{5}\ell^4 \end{bmatrix}$$

whence :

$$I_2^*(L_o) = \left((I_2(L_o))^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta\alpha^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1}$$

$$I_2'(L_o) = \begin{bmatrix} 1 & 0 & 0 \\ L_o & 1 & 0 \\ \frac{L_o^2}{2} & L_o & 1 \end{bmatrix} I_2^*(L_o) \begin{bmatrix} 1 & L_o & \frac{L_o^2}{2} \\ 0 & 1 & L_o \\ 0 & 0 & 1 \end{bmatrix}$$

and :

$$I_1(0) = \frac{1}{\sigma^2} \begin{bmatrix} n_o & \frac{n_o}{2}\ell & \frac{n_o}{3}\ell^2 \\ \dots & \frac{n_o}{3}\ell^2 & \frac{n_o^4}{4}\ell^3 \\ \dots & \dots & \frac{n_o^5}{5}\ell^4 \end{bmatrix}$$

We do not calculate explicity $I_o = I_1 + I_2'$

For $n \to \infty$ one finds :

$$V_{cc}^{\infty} = 0 \quad ; \quad V_{YY}^{\infty} = L_o^2 \, V_{aa}^{\infty} = \frac{\sigma^2}{n_o (\frac{1}{\rho^2} + \frac{1}{3})} \tag{4.6}$$

i.e. the same limits as for Z and b in (4.5).

This can be understood in the following way : with an infinite measured length the quadratic term is known without uncertainty from the second portion ; then it can be subtracted without information loss and the fitting procedure is reduced to determine Y and a in the same conditions as Z and b previously.

The comparison between the optimal fit and the standard one is illustrated on fig. 2. For $\rho > 16$ the gain on Y and a may be large w.r.t. the minimum of the standard fit, if the measurements do not extend up to the second minimum ; the gain on c is large in a wide range, although it vanishes for $n \to \infty$.

4.3. Uniformly distributed scattering :

4.3.1. Principle of the calculation :

Thanks to the formalism developed in 3.1., we can give a recursive expression of the optimal information matrix. Let I_n be the matrix obtained with n points, for the parameters Z, b or Y, a, $\frac{c}{2}$, at the first measured point ; let V_n be its inverse, i.e. the covariance matrix of the optimal fitted parameters. If the track has n + 1 measured points, the last n points give the information matrix I_n for the parameters at the second point ; then the first point can be added within a step of the recursion : so we obtain I_{n+1} for the parameters at the first point.

4.3.2. Straigtht line fit :

The formulae analogous to (3.2) and (3.5) give in this case :

$$I_{n+1} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \left((I_n)^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & \delta\alpha^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.7}$$

$$= \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \left(I_n - \frac{\delta\alpha^2}{\delta\alpha^2 + (I_n)_{bb}} \begin{bmatrix} (I_n)^2_{Zb} & (I_n)_{Zb}(I_n)_{bb} \\ \cdots & (I_n)^2_{bb} \end{bmatrix} \right) \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.8}$$

This expression suggests an expansion of I_n in powers of $\delta\alpha^2$; at order zero we find :

$$I_{n+1} = \begin{bmatrix} 1 & 0 \\ \ell & 0 \end{bmatrix} I_n \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{bmatrix}$$

i.e.
$$\left.\begin{aligned} (I_{n+1})_{ZZ} &= (I_n)_{ZZ} + \frac{1}{\sigma^2} \\ (I_{n+1})_{Zb} &= (I_n)_{Zb} + \ell\,(I_n)_{ZZ} \\ (I_{n+1})_{bb} &= (I_n)_{bb} + 2\ell\,(I_n)_{Zb} + \ell^2 (I_n)_{ZZ} \end{aligned}\right\} \tag{4.9}$$

This gives the usual information matrix in the absence of scattering :

$$I_n^{(0)} = \frac{1}{\sigma^2} \begin{bmatrix} n & \frac{n^2\ell}{2} \\ \cdots & \frac{n^3\ell^2}{3} \end{bmatrix} \qquad \text{(cf. (2.3))}$$

Then we expand (4.8) at first order in $\delta\alpha^2$, replacing I_n by $I_n^{(0)}$ in the $\delta\alpha^2$ terms. So we obtain the recursive relations satisfied by the first order approximation $I_n^{(1)}$:

$$(I_{n+1}^{(1)} - I_n^{(1)})_{ZZ} = \frac{1}{\sigma^2} - \delta\alpha^2\,(I_n^{(0)})_{ZZ}$$

$$(I_{n+1}^{(1)} - I_{n+1}^{(1)})_{Zb} = \ell\,(I_n^{(1)})_{ZZ} - \delta\alpha^2\,(I_n^{(0)})_{Zb}\left[\ell(I_n^{(0)})_{Zb} + (I_n^{(0)})_{bb}\right]$$

$$(I_{n+1} - I_{n+1})_{bb} = \ell^2 (I_n^{(1)})_{ZZ} + 2\ell\,(I_n^{(1)})_{Zb} - \delta\alpha^2\left[\ell(I_n^{(0)})_{Zb} + (I_n^{(0)})_{bb}\right]^2$$

These relations give successively :

$$(I_n^{(1)})_{ZZ} = \frac{1}{\sigma^2}\left(n - q^2\,\frac{n^5}{20}\right)$$

$$(I_n^{(1)})_{Zb} = \frac{\ell}{\sigma^2}\left(\frac{n^2}{2} - \frac{13}{360}\,q^2\,n^6\right)$$

$$(I_n^{(1)})_{bb} = \frac{\ell}{\sigma^2}\left(\frac{n^3}{3} - \frac{11}{420}\,q^2\,n^7\right)$$

or, with $\varepsilon = q^2 n^4$

$$I_n^{(1)} = \frac{1}{\sigma^2} \begin{bmatrix} n\,(1 - \frac{\varepsilon}{20}) & \frac{n^2\ell}{2}\,(1 - \frac{13\varepsilon}{180}) \\ \cdots & \frac{n^3\ell^2}{3}\,(1 - \frac{11\varepsilon}{140}) \end{bmatrix} \qquad (4.10)$$

This approximation is valid so long as ε terms in (4.10) are small compared to 1, what can be summarized by $\varepsilon << 20$ (of course this implies $q^2 << 1$).

Since ε is proportional to n^4, (4.10) remains a good approximation of I_n up to $n \simeq \frac{1}{\sqrt{q}}$.

By inverting $I_n^{(1)}$ one finds the approximate covariance matrix :

$$V_n^{(1)} = \sigma^2 \begin{bmatrix} \frac{1}{n}\,(4 + \frac{\varepsilon}{105}) & -\frac{1}{n^2\ell}\,(6 + \frac{11\varepsilon}{210}) \\ \cdots & \frac{1}{n^3\ell^2}\,(12 + \frac{13\varepsilon}{35}) \end{bmatrix} \qquad (4.11)$$

namely the covariance of the standard estimators (cf (2.11)). Thus the <u>standard</u> fit is <u>quasi-optimal</u> up to $n \simeq \frac{1}{\sqrt{q}}$, and it is useless to search some other estimators (e.g. giving to the measured points weights decreasing with n).

For higher values of n, we know only that $(V_n)_{ZZ}$ and $(V_n)_{bb}$ must decrease and tend towards finite limits ; likely, $(V_n)_{Zb}$ too. So we expect the matrix V_n to have also a limit V for $n \to \infty$. According to (4.8), V should verify :

$$V^{-1} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix} \left(V + \begin{bmatrix} 0 & 0 \\ 0 & \delta\alpha^2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.12)$$

this can also be written as :

$$\left(V^{-1} - \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & -\ell \\ 0 & 1 \end{bmatrix} \left(V + \begin{bmatrix} 0 & 0 \\ 0 & \delta\alpha^2 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix}$$

or :

$$V + \frac{1/\sigma^2}{1 + V_{ZZ}/\sigma^2} \begin{bmatrix} V_{ZZ}^2 & V_{ZZ}V_{Zb} \\ \cdots & V_{Zb}^2 \end{bmatrix} = \begin{bmatrix} V_{ZZ} - 2\ell V_{Zb} + \ell^2 V_{bb} & V_{Zb} - \ell V_{bb} \\ \cdots & V_{bb} + \delta\alpha^2 \end{bmatrix} \tag{4.13}$$

Hence a system of three algebraic equations in V_{ZZ}, V_{Zb} and V_{bb}, which becomes with

$$u = \frac{V_{ZZ}}{\sigma^2} \quad , \quad v = \frac{\ell\, V_{Zb}}{\sigma^2} \quad , \quad w = \frac{\ell^2 V_{bb}}{\sigma^2} \quad :$$

$$\begin{cases} u^4 = q^2 (1-u)(2-u)^2 \\ v^2 = q^2 (1-u) \\ w = q^2 (1+\frac{u}{v}) \end{cases}$$

If q is small, the solutions can be expanded in powers of $\sqrt{q}$:

$$\begin{aligned} V_{ZZ} &= \sqrt{2\,\sigma^3\,\ell\,\delta\alpha} \quad \left(1 - \sqrt{\frac{q}{2}} - \frac{5}{8} q + \ldots \right) \\ V_{Zb} &= -\sigma\,\delta\alpha \quad \left(1 - \sqrt{\frac{q}{2}} - \frac{3}{4} q + \ldots \right) \\ V_{bb} &= \sqrt{\frac{2\,\sigma\,\delta\alpha^3}{\ell}} \quad \left(1 - \sqrt{\frac{q}{2}} + \frac{1}{8} q + \ldots \right) \end{aligned} \tag{4.14}$$

Fig. 3 shows the comparison between the standard uncertainties and the optimal ones : the gain remains moderate as far as the measured length is less than the optimal length defined in 2.3 for the standard estimator. It is interesting to compare the limits previously found to the minima of the standard variances, which were, according to (2.12) :

$$V_{ZZ}^{min} = 1.550 \sqrt{\sigma^3\,\ell\,\delta\alpha}$$

$$V_{bb}^{min} = 1.554 \sqrt{\frac{\sigma\,\delta\alpha^3}{\ell}}$$

Thus, for $q \ll 1$, the gain of the optimal fit with respect to the standard fit applied to the truncated track, remains always limited, whatever the measured length.

4.3.3. Parabolic fit

With the parameters Y, a and $\frac{c}{2}$ the recursion relation becomes :

$$I_{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ \ell & 1 & 0 \\ \frac{\ell^2}{2} & \ell & 0 \end{bmatrix} \left(I_n^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta\alpha^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \ell & \frac{\ell^2}{2} \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.15}$$

The first order expression $V_n^{(1)}$ for small n is again equal to the covariance matrix of the standard estimator, which is then quasi-optimal up to $n \simeq \frac{1}{\sqrt{q}}$

For $n \to \infty$, $(V_n)_{cc}$ must tend to zero, since the variance of the standard estimator already vanished. Thus, for the same reasons as discussed above (cf. 4.2.3.), $(V_n)_{YY}$, $(V_n)_{aa}$ and $(V_n)_{Ya}$ have the same limits as $(V_n)_{ZZ}$, $(V_n)_{bb}$ and $(V_n)_{Zb}$ respectively (given by (4.14)).

The minimal variances of the standard estimator are here :

$$V_{YY}^{min} = 1.820 \sqrt{\sigma^3 \ell \delta\alpha}$$

$$V_{aa}^{min} = 2.159 \sqrt{\frac{\sigma \delta\alpha^3}{\ell}}$$

This shows that the gain of the optimal fit w.r.t. the "truncated" standard fit is potentially bigger than without curvature. However, for a given value if q, $(V_n)_{YY}$ and $(V_n)_{aa}$ converge much less rapidly than $(V_n)_{ZZ}$ and $(V_n)_{aa}$: so the actual gain is generally small, as it can be seen on fig. 4.

5. IMPLEMENTATION OF THE RECURSIVE METHOD :

5.1. Description of the experiment :

We used real events from the C.E.R.N. WA13 experiment, which was realized in the Omega spectrometer equipped with optical spark chambers ; its aim was to study hadronic large-angle reactions giving in the final state two charged particles or two neutrals decaying in charged mode ; detailed description of the experiment can be found in (3) and (4). Fig. 4 summarizes the measurement apparatus: in their measured section, the tracks crossed mainly the spark chambers (0.06 L_{rad} for the whole set of chambers) and a 1 cm thick scintillator S6.

We used events produced at 3 GeV/c, and we did not apply kinematical cuts : the momenta of the reconstructed tracks ranged from a few hundred MeV/c ; the outgoing particles were not identified.

5.2. Programmation of the algorithm :

The recursive method was programmed in the following way :

- All matrix operations were written explicitly, without subscripted variables or loops, in order to save computation time (this is possible because all matrices involved have the same dimension).

- The propagation between two points (generally separated by 6 to 10 cm) was realized in two steps by a fourth-order Taylor expansion of the trajectory ; the propagation of the information matrix used the second order approximation (cf. (3.13)). The magnetic field was assumed to be uniform between these two points. In most cases this calculation was much too accurate ; it was needed for low-momentum particles emitted at large angle.

Every time the distance between two measured points exceeded 10 cm, dummy points were inserted.

- The variances of the scattering angles were calculated in two velocity hypothesis : ultrarelativistic (valid for π) and proton.

- The measured points were extracted with their weights defined at fixed x, from ROMEO (standard reconstruction program for Omega events), just before the final fit, so that both fitting procedures worked on exactly the same input.

- The starting value of the curvature parameter was estimated with three points (at the beginning, in the middle and at the end of the track) ; for the slope parameters, the last two points were used, with a curvature correction. As a matter of fact, the fitted values do not appear to depend stongly on the initial ones. After 3 or 4 recursion steps, the gaps between the extrapolated trajectory and the measured points are compatible with the measurements errors, and moderate deviations on the starting values are automatically corrected : for example, estimating the slope without curvature correction between the last two points has no appreciable effect on the fitted values.

The computation time for one recursion step (propagation + measurement + scattering) amounted to about 360 μs on CDC CYBER 750, equivalent to 160 μs on CDC 7600 (the computation of the magnetic field is not included). The time for a whole track is obviously proportional to the number of points on this track.

5.3. Test of the precision :

From the parameters fitted at the first measured point we extrapolated the tracks backwards to find intersections, and we determined the points of closest approach between each pair of tracks with opposite sign.

Fig. 5 shows the distributions of the minimal distances obtained with the standard program ROMEO, and with our algorithm (for both mass hypotheses).

The precision is clearly improved by our method. The improvement is not very strong, but anyway the theoretical gain that we could deduce from sect. 4 for this experiment, is not enormous. Moreover there was matter accumulated around the vertex (the hydrogen target was surrounded by scintillators) which worsened its geometrical precision, and for which optimization cannot help. For the same reason, no appreciable improvement can be expected in the kinematical resolution.

6. CONCLUSION

The standard fitting procedures, applied to slow tracks measured over a big length, and/or with high accuracy, lead to important losses of precision on the geometrical parameters. They can be improved by taking account of the measurements up to a certain length only ; this optimal length can be evaluated from analytical approximations of the variances. In many cases, especially in a homogeneous medium, this "truncated" fit is not far from being optimal. Its main drawback is that the optimal length depends on the choice of parameter for which one wants the highest precision ; moreover it was calculated assuming gaussian independent measurement errors, and we do not know how it would be modified by small uncorrected geometrical distorsions, or by non-gaussian errors (e.g. in a M.W.P.C.).

The optimal fit (without information loss) is usually realized by calculating the whole covariance matrix of the measurements, or by adding extra parameters to describe the scattering (1) ; these methods involve handling big matrices. In this paper we described a new implementation of the optimal estimator : the parameters are fitted backwards by introducing the measured points one at a time from the end to the beginning of the track. The elementary steps are thus : including one or several raw measurements, and/or one elementary scattering : from one point to the next one the parameters and their information (or weight) matrix are propagated by a local polynominal parametrization.

Such a procedure is very flexible : it can be applied to various types of detectors, and especially to composite detectors. Moreover it does not require more computation time than the standard ones (perhaps less) ; contrary to the "big matrix" optimal methods, the time used is merely proportional to the number of points.

The implementation of this algorithm on real events shows an improvement of the geometrical accuracy ; its order of magnitude corresponds to what could be expected. The fitted values of the parameters appear to be stable with respect to the starting value chosen at the end of the track : thus no iteration is needed to obtain the best estimate ; the algorithm stabilizes after 3 or 4 steps along the track.

Finally we point out that this method, in some configurations, could be able to perform at the same time the pattern recognition and the geometrical reconstruction of the tracks, since it gives in a short time the best extrapolation of a track candidate built up with the most external points.

I thank Professor M. Froissart for helpful discussions and suggestions.

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FIGURE CAPTIONS

==========

Fig. 1 to 4 :

Errors on the track parameters (position and slope at first point, and eventually curvature) as functions of the measured length for both standard and optimal estimators :

1. straight track
2. curved track

} scattered at one point

3. straight track
4. curved track

} with uniformly distributed scattering

Fig. 5 :

Layout of the WA13 experiment in the Omega Spectrometer.

Fig. 6 :

Comparison between the standard fitting method of ROMEO and the recursive implementation of the optimal estimator : distance, at the point of closest approach, between two tracks extrapolated backwards to the vertex.

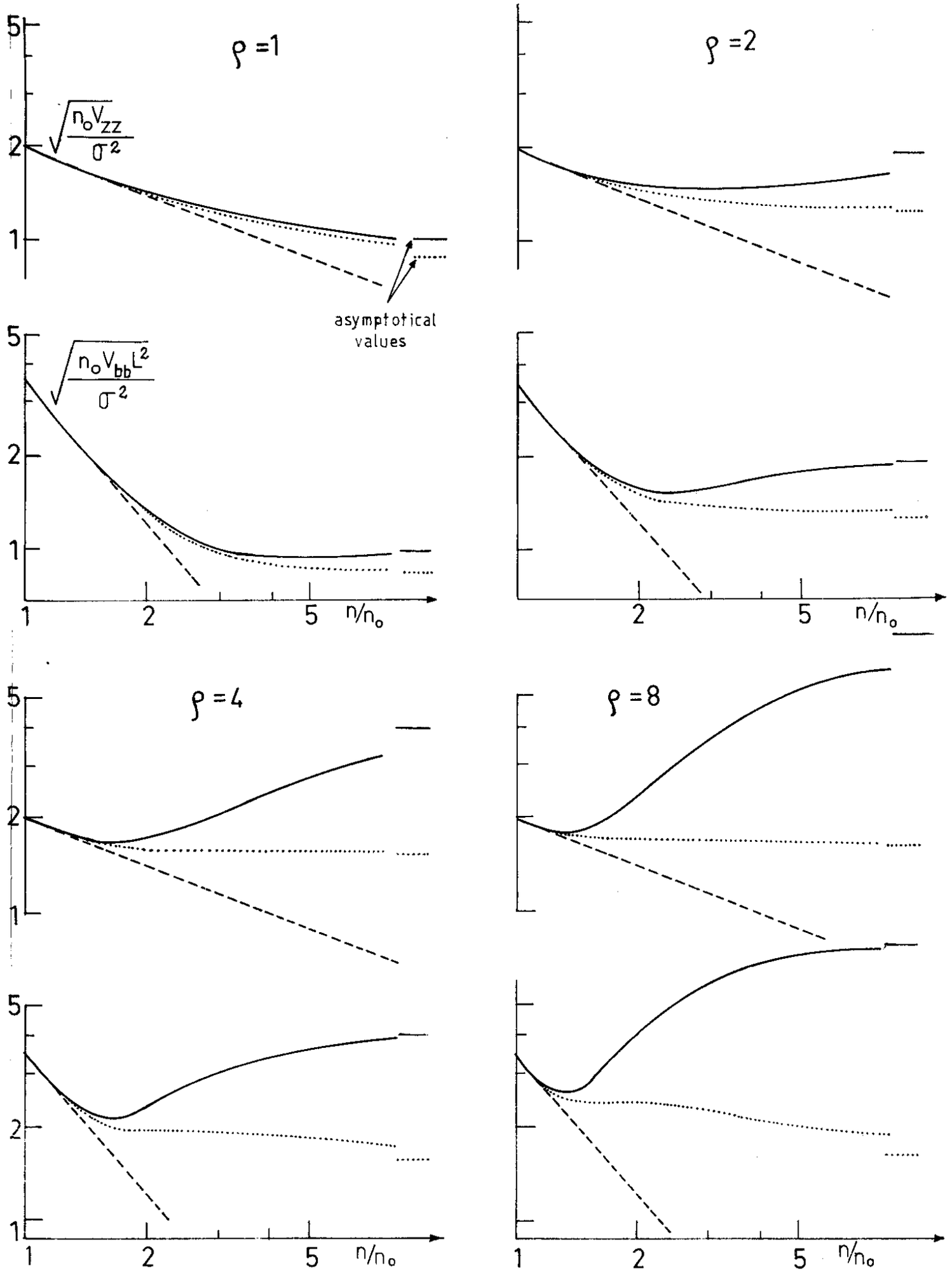

ρ =1
ρ =2
ρ =4
ρ =8
$\sqrt{\frac{n_o V_{zz}}{\sigma^2}}$
$\sqrt{\frac{n_o V_{bb} L^2}{\sigma^2}}$
asymptotical values
n/n_o
standard fit
optimal fit
with scattering
without scattering

FIG. 1

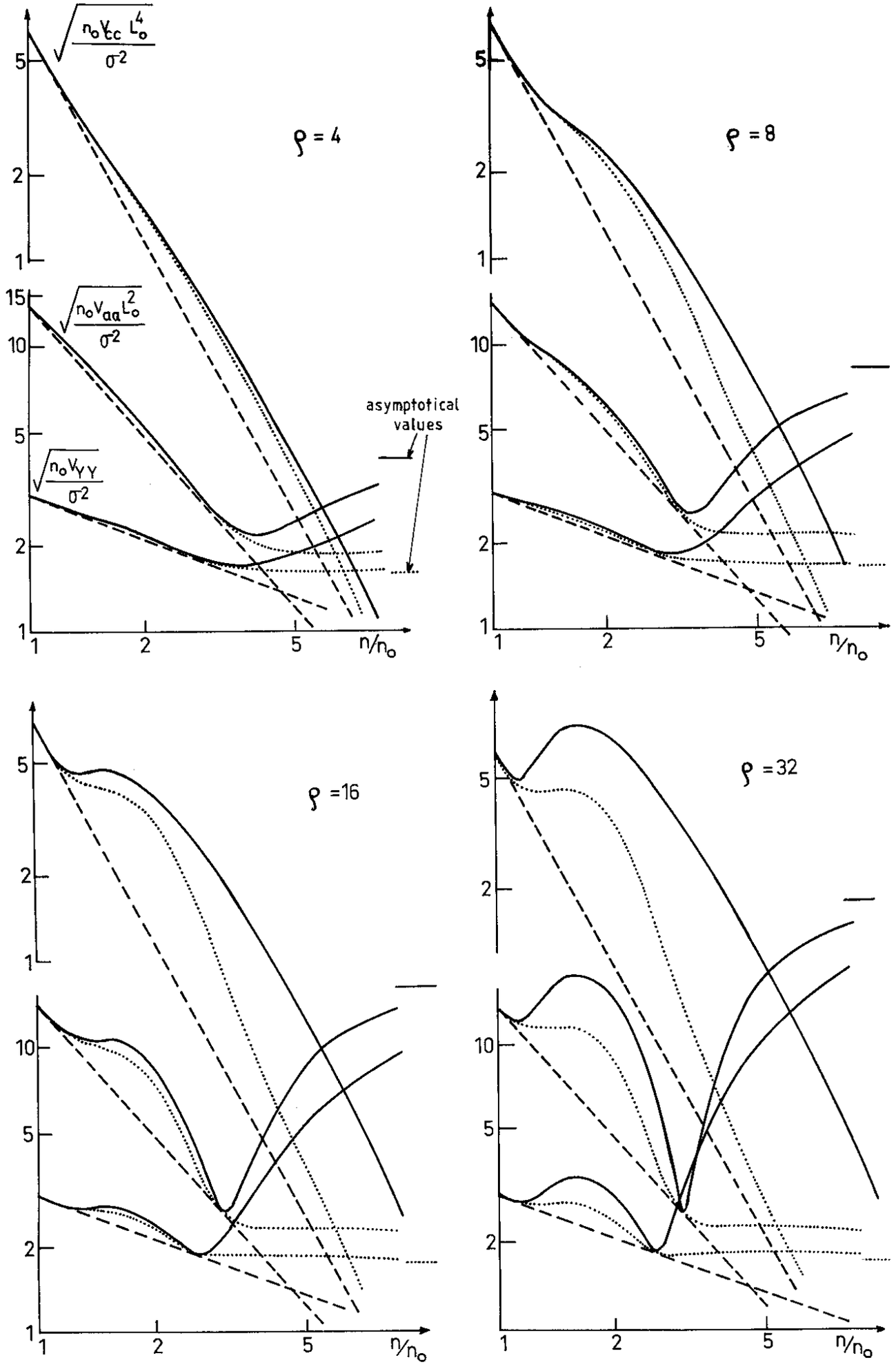

$\sqrt{\frac{n_o V_{cc} L_o^4}{\sigma^2}}$
$\rho = 4$
$\rho = 8$
$\sqrt{\frac{n_o V_{aa} L_o^2}{\sigma^2}}$
$\sqrt{\frac{n_o V_{YY}}{\sigma^2}}$
asymptotical values
$\rho = 16$
$\rho = 32$
n/n_o

FIG. 2

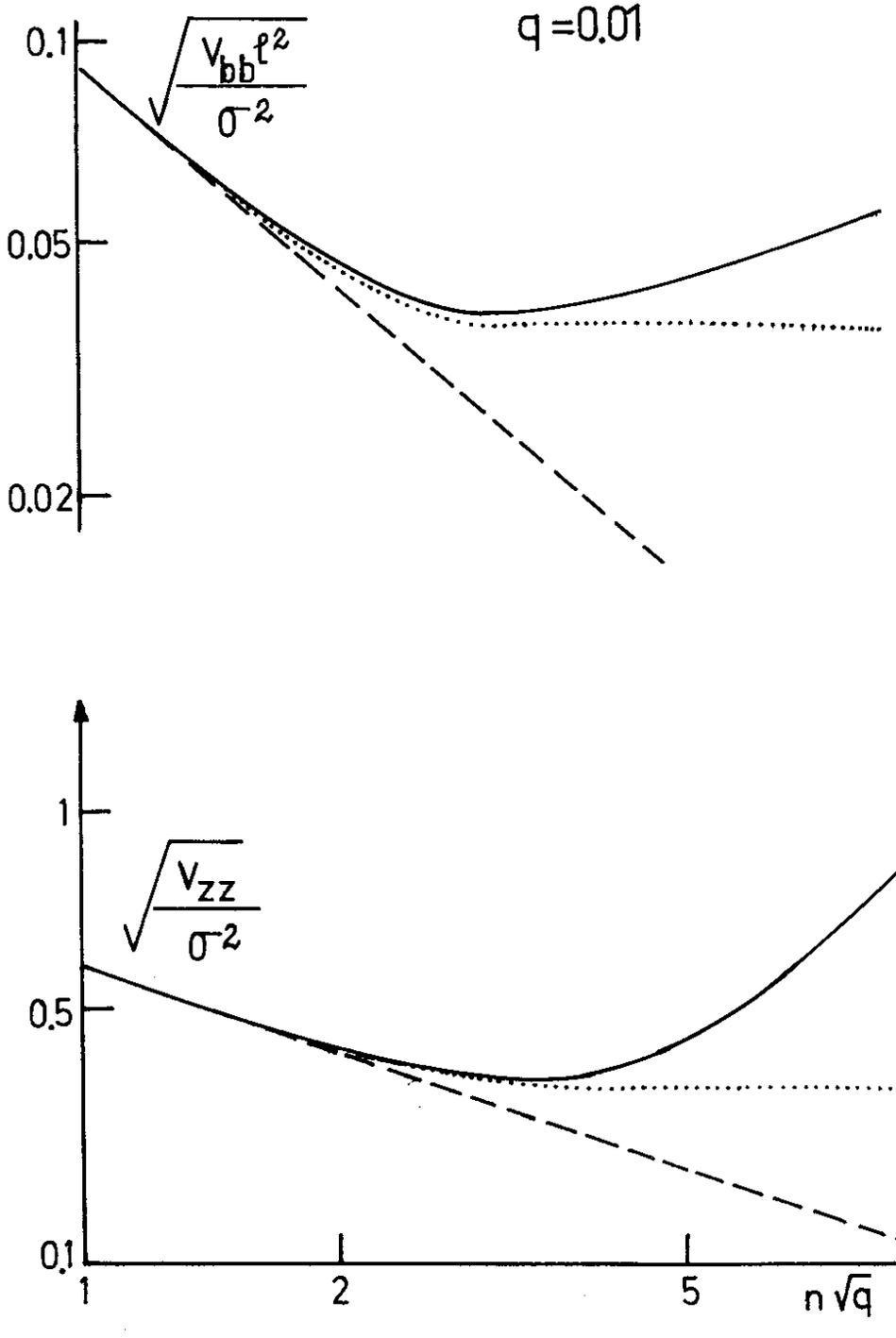

q =0.01
$\sqrt{\frac{V_{bb}\ell^2}{\sigma^2}}$
0.1
0.05
0.02
$\sqrt{\frac{V_{zz}}{\sigma^2}}$
1
0.5
0.1
1
2
5
n√q

FIG. 3

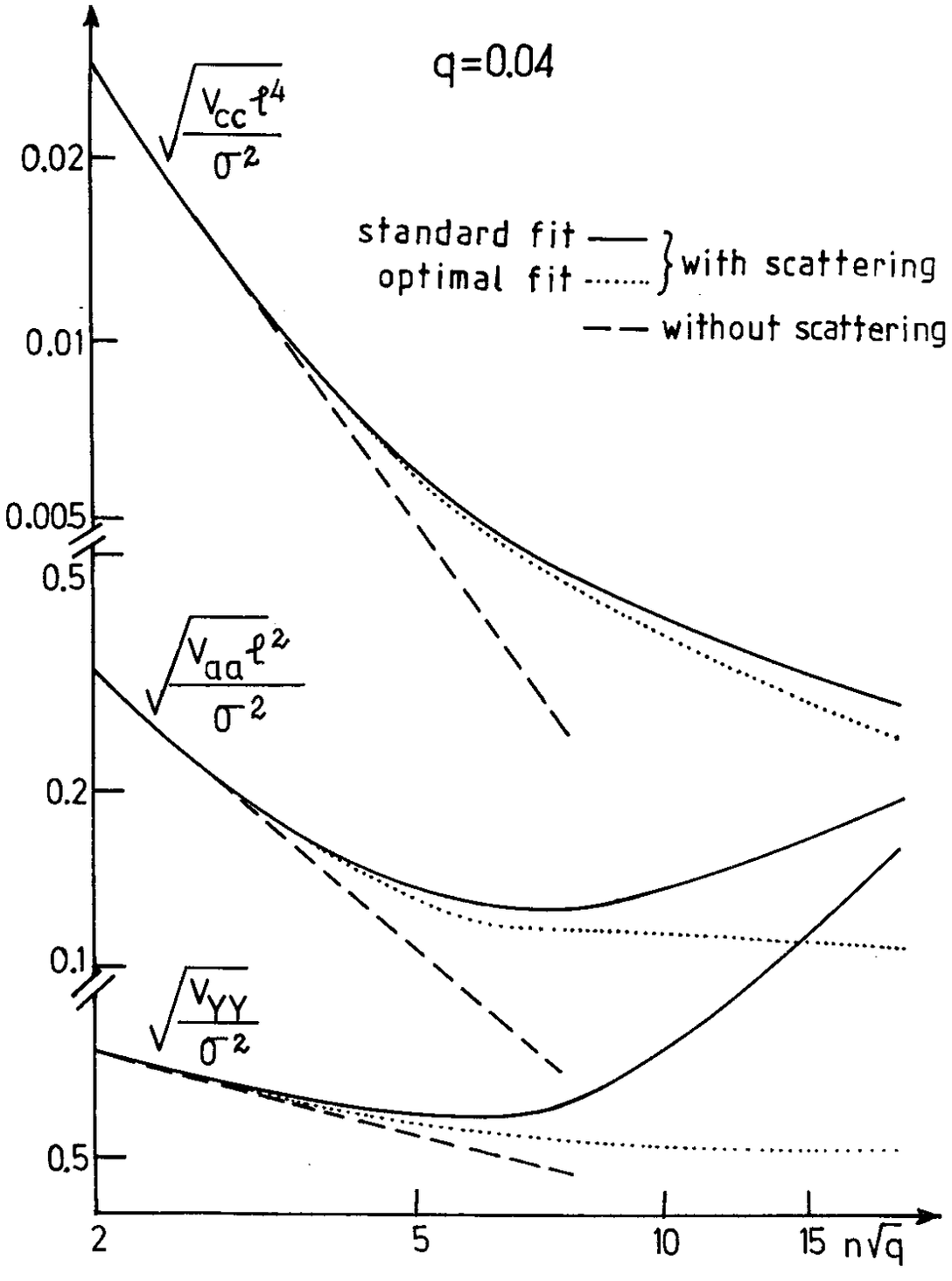

q=0.04
$\sqrt{\frac{V_{cc}\ell^4}{\sigma^2}}$
standard fit
optimal fit
with scattering
without scattering
0.02
0.01
0.005
0.5
$\sqrt{\frac{V_{aa}\ell^2}{\sigma^2}}$
0.2
0.1
$\sqrt{\frac{V_{YY}}{\sigma^2}}$
0.5
2
5
10
15 n√q

FIG. 4

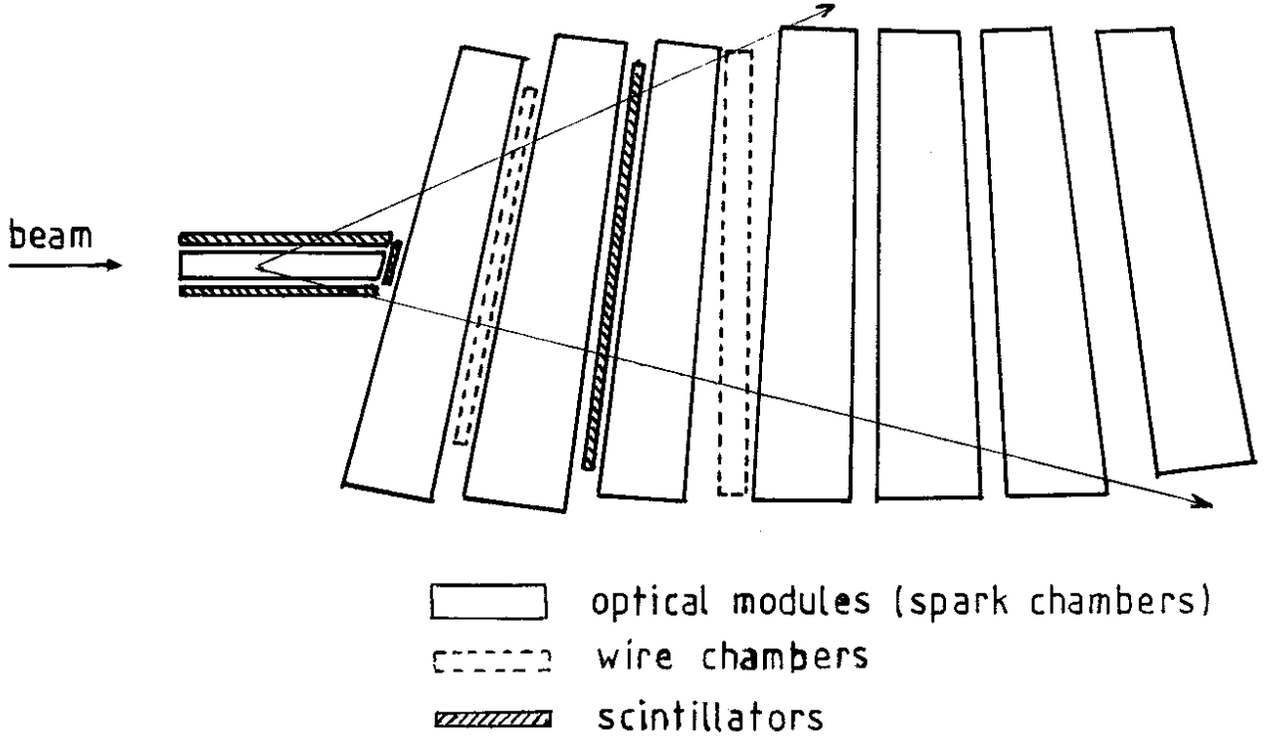

beam
optical modules (spark chambers)
wire chambers
scintillators

FIG. 5

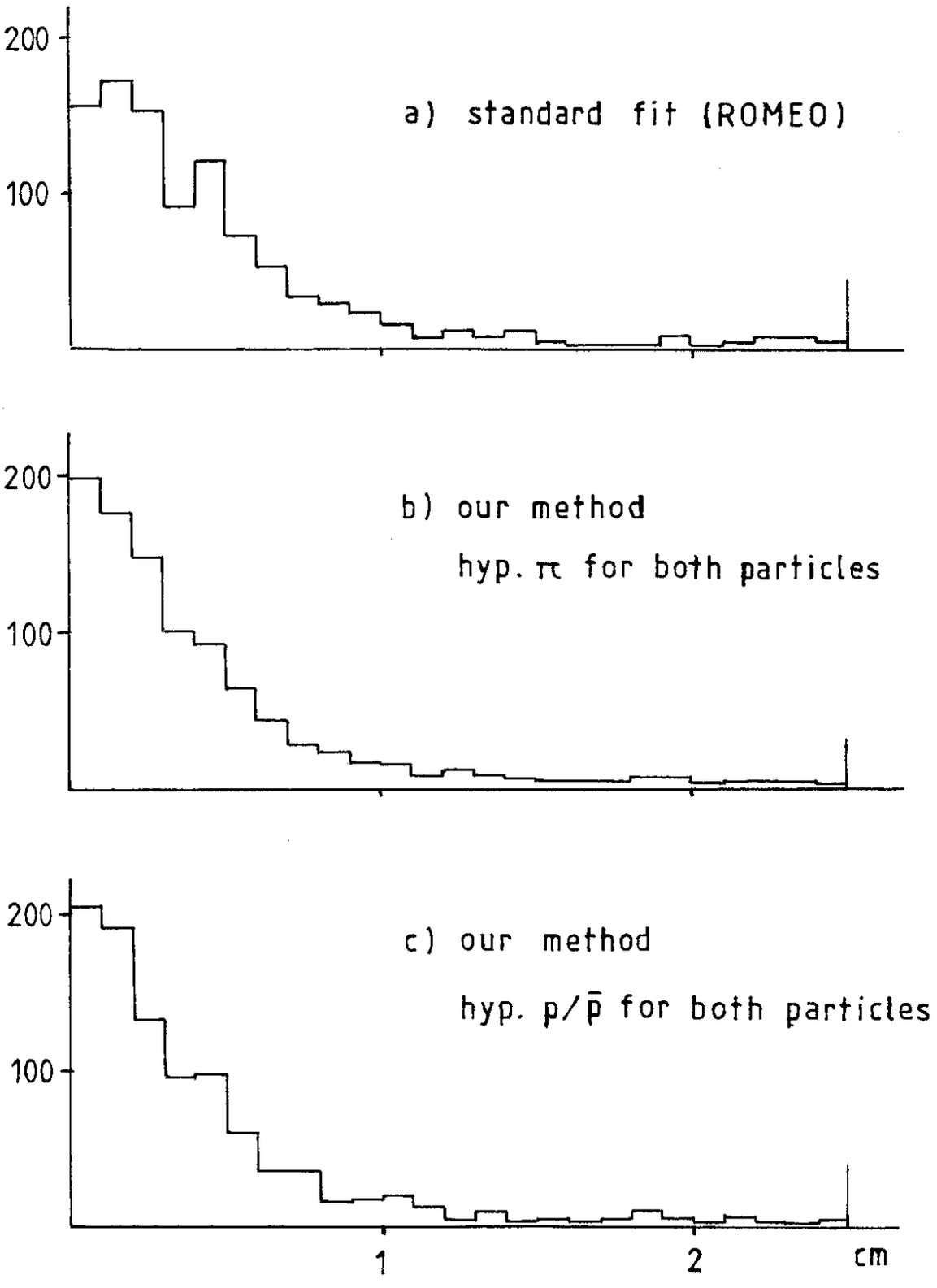
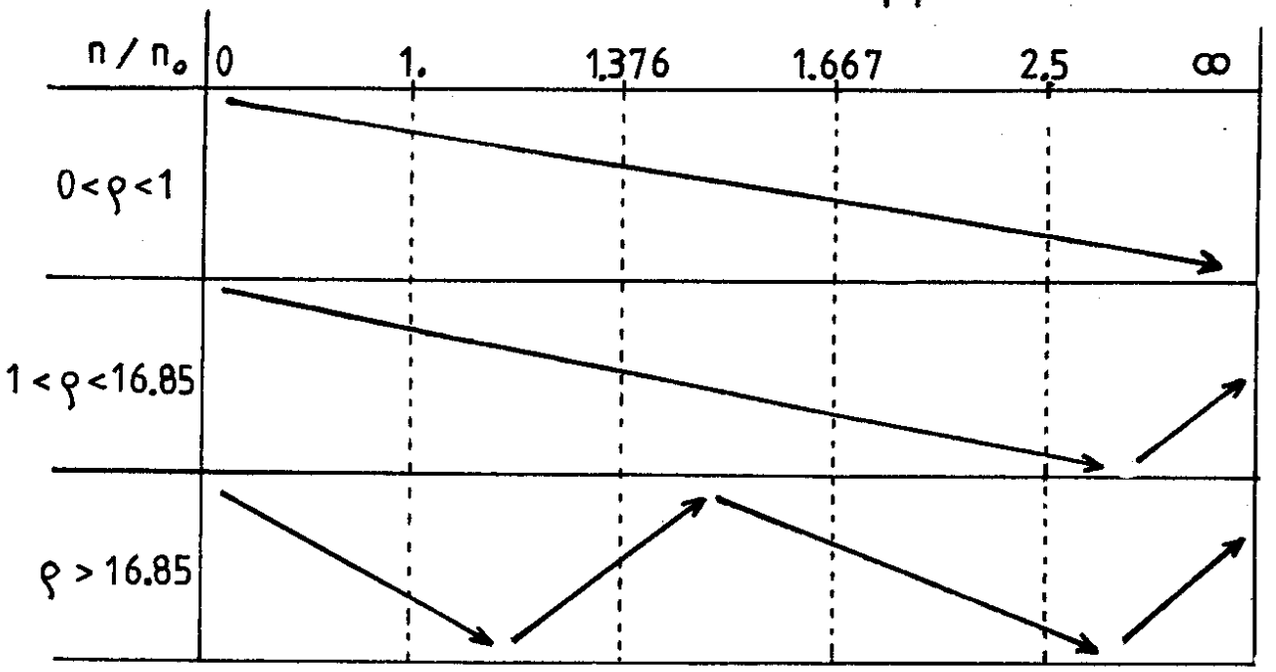

200
100
a) standard fit (ROMEO)
200
100
b) our method
hyp. π for both particles
200
100
c) our method
hyp. p/p̄ for both particles
1
2
cm

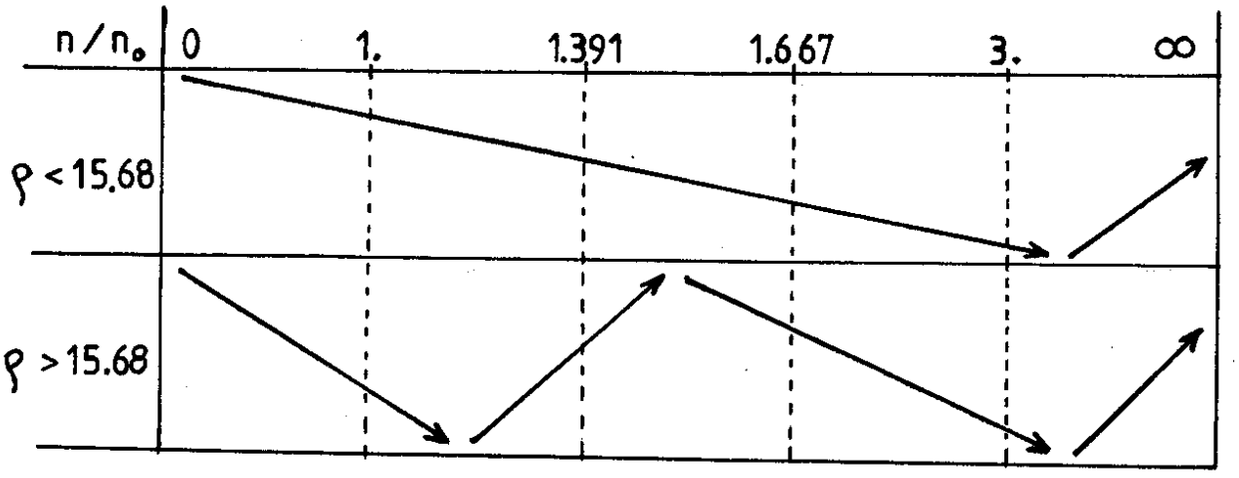
FIG. 6

TABLE I : VARIANCES OF THE STANDARD ESTIMATORS

variation of V_{YY}



variation of V_{aa}



variation of V_{cc}

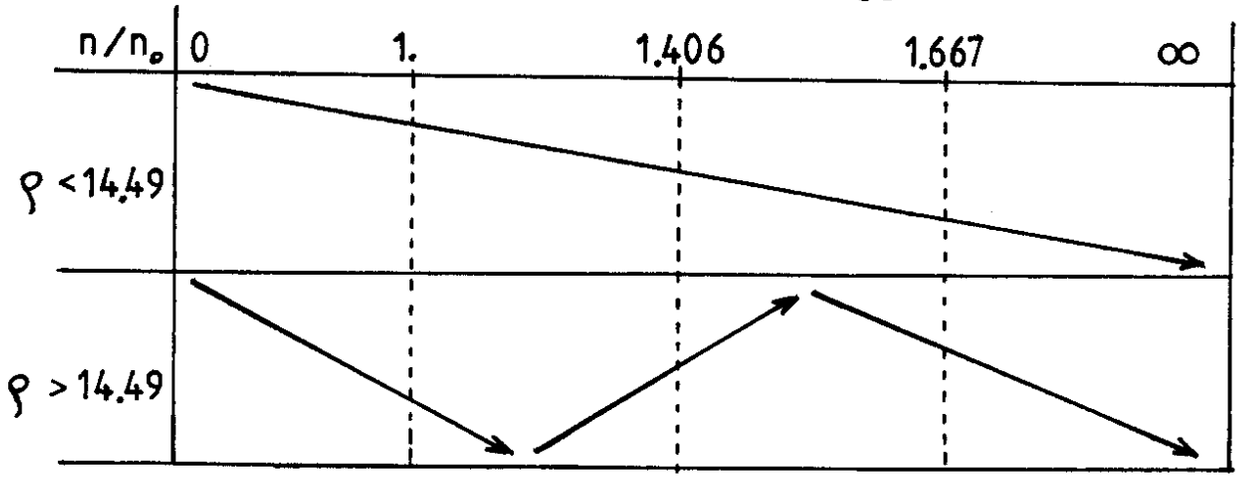


TABLE 2 : ONE STEP OF THE RECURSIVE ESTIMATOR

	Estimator	Information matrix
	at x_n : y_n^{opt} a_n^{opt} c_n^{opt}	I_n
scattering at x_n	↓	↓
	at x_n : y_n^{opt} a_n^{opt} c_n^{opt}	$I_n^* = (I_n^{-1} + A_n)^{-1}$
propagation from x_n to x_{n-1}	↓	↓
	at x_{n-1} : y'_{n-1} a'_{n-1} c'_{n-1} given by : $\begin{bmatrix} y'_{n-1} \\ a'_{n-1} \\ c'_{n-1} \end{bmatrix} = D_n \begin{bmatrix} y_n^{opt} \\ a_n^{opt} \\ c_n^{opt} \end{bmatrix}$	$I'_{n-1} = (D_n^t)^{-1} I_n^* (D_n)^{-1}$
measurement at x_{n-1}	↓	↓
	at x_{n-1} : y_{n-1}^{opt} a_{n-1}^{opt} c_{n-1}^{opt} given by the equation : $(I'_{n-1}+M_n) \begin{bmatrix} y_{n-1}^{opt}-y'_{n-1} \\ a_{n-1}^{opt}-a'_{n-1} \\ c_{n-1}^{opt}-c'_{n-1} \end{bmatrix} = M_n \begin{bmatrix} y_{n-1}^{m}-y'_{n-1} \\ 0 \\ 0 \end{bmatrix}$	$I_{n-1} = I'_{n-1} + M_n$

In the parabolical schematization :

$$D_n = \begin{bmatrix} 1 & -\delta x_n & \delta x_n^2/2 \\ 0 & 1 & -\delta x_n \\ 0 & 0 & 1 \end{bmatrix} \qquad A_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta\alpha_n^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_n = \begin{bmatrix} 1/\sigma^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$